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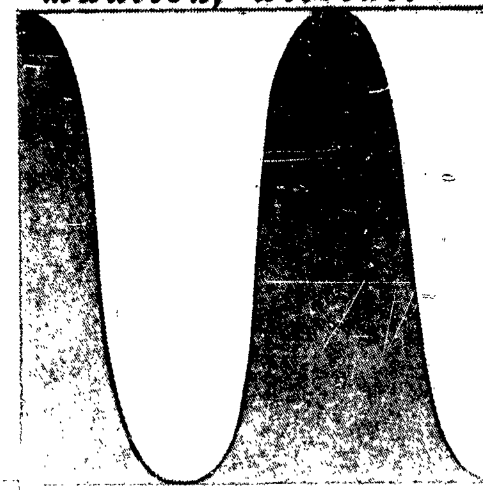

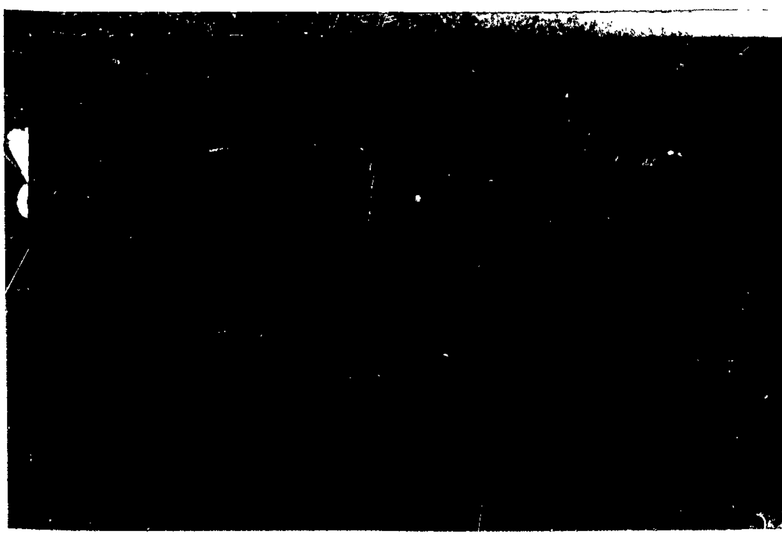

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SOME FAMILIES OF COMPOUND AND GENERALIZED DISTRIBUTIONS

John Gurland

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ABSTRACT

Compound and generalized distributions have been discussed in the framework of contagious distributions. In particular, it is pointed out that the Negative Binomial may be regarded as a compound Poisson (using a Gamma variable as the compounder) or as a generalized Poisson (using a Logarithmic random variable as the generalizer). As an example of true contagion the Negative Binomial is also a limit of the distribution obtained through Polya's urn model.

A formal relation between compound and generalized distributions is developed, utilizing a symbolic notation. Some natural extensions of the Negative Binomial through repeated compounding with a Gamma distribution or through repeated generalizing with a Logarithmic distribution are indicated.

Some wide generalizations of Neyman's class of contagious distributions are presented, and examination of their shape reveals that some simpler families with fewer parameters, such as the Poisson v Pascal offer interesting possibilities for fitting data. An attractive property of the Poisson v Pascal is that it contains the Negative Binomial, Neyman Type A, and Poisson as special limiting cases.

SOME FAMILIES OF COMPOUND AND GENERALIZED DISTRIBUTIONS

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1. Introduction

Compound and generalized distributions arise in the study of so-called contagious distributions. Feller (1943) described two types of contagion. One of these types, "true contagion", pertains to situations in which each "favorable" event increases (or decreases) the probability of succeeding favorable events. The other of these types, "apparent contagion", reflects a sort of heterogeneity of the population. Still a further type of contagion known as a "model of random colonies" also proves useful in the study of many biological phenomena. This type of contagion is described by means of generalized distributions.

The main purpose of this paper is an expository presentation of some results on contagious distributions in which the relation between a certain class of compound and of generalized distributions is utilized. Some general families of contagious distributions are described and their shape characteristics indicated. Some consideration is also given as to the selection of an appropriate family of distributions when one is attempting to fit data on the basis of an underlying model of the type described here.

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2. Contagion

2.1 Apparent contagion

This type of contagion is the result of a mixture of distributions arising through the distribution of a parameter in an initial distribution. A well known example is the result of applying a Gamma distribution to the mean of a Poisson distribution. (cf. Greenwood and Yule (1920)). Specifically, let the mean of the initial distribution (the Poisson, in this example) be λ . The probability generating function (p.g.f.) of this Poisson distribution is

$$e^{\lambda(z-1)} \quad . \tag{1}$$

By the p.g.f. $g(z)$ of a random variable X we mean Ez^X , where E denotes expectation. When the values which X may assume (with non-zero probability) are non-negative integers then the p.g.f. expressed as a power series yields the probabilities as the coefficients in the series. Thus

$$g(z) = \sum_{r=0}^{\infty} z^r P\{X = r\} \quad . \tag{2}$$

On applying a Gamma distribution with probability density

$$p(\lambda) = \frac{\alpha^{\beta}}{\Gamma(\beta)} e^{-\alpha\lambda} \lambda^{\beta-1} \qquad \begin{array}{l} \lambda > 0 \\ \alpha > 0 \\ \beta > 0 \end{array} \tag{3}$$

to the mean λ in the above initial Poisson distribution we obtain for the p.g.f. of the resulting distribution

$$\frac{\alpha^{\beta}}{\Gamma(\beta)} \int_0^{\infty} e^{-\alpha\lambda} \lambda^{\beta-1} e^{\lambda(z-1)} d\lambda = (1 - \frac{z-1}{\alpha})^{-\beta} \quad . \tag{4}$$

If we write $p = \frac{1}{\alpha}$; $q = 1 + p$; $\beta = k$, the p.g.f. in (4) becomes

$$(q-pz)^{-k} \tag{5}$$

which is a well known form for the p.g.f. of a Negative Binomial distribution. This is an example of apparent contagion, and on the basis of this model the Negative Binomial may be regarded as a compound Poisson distribution. A formal definition of a compound distribution will be given in section 3.

2.2 True contagion

The following urn scheme due to Polya (1930) affords an example of true contagion and leads in a relatively simple way to the Negative Binomial distribution. Let an urn contain Np white and Nq black balls, where $p + q = 1$. Suppose n successive drawings of a ball are made according to the following rule: If a white ball is drawn it is replaced and $N\delta$ additional white balls are put in the urn. Likewise, if a black ball is drawn it is replaced and $N\delta$ additional black balls are put in the urn.

Polya (op. cit.) shows that when $p \to 0$, $\delta \to 0$, $n \to \infty$ such that np and $n\delta$ are held constant the distribution of the number of white balls approaches that of a Negative Binomial random variable. It is a fact of considerable interest (cf. Arbous and Kerrich (1951); Fitzpatrick (1958)) that the Negative Binomial may be regarded as arising through apparent contagion or through true contagion.

2.3 Model of random colonies

This model has wide application in biological as well as other phenomena. An example illustrating the mechanism of this model is afforded by the distribution of insects over a field. Suppose the insects are larvae which hatched from egg-masses. These egg-masses may be regarded as cluster centers or "random colonies". Actually two underlying distributions are involved in the final distribution of the larvae. First, there is the distribution of the egg-masses over the field; second, there is the distribution of larvae which leave an egg-mass and arrive at a particular location selected at random.

Specifically, let the distribution of egg-masses be Poisson with p.g.f.

$$g_1(z) = e^{\lambda(z-1)} = P_0 + P_1 z + P_2 z^2 + \dots \qquad (6)$$

where $P_r = e^{-\lambda} \lambda^r / r!$ is the probability that exactly r egg-masses are represented on a randomly selected location. Suppose, further, that the number of larvae from an egg-mass which reach the location is given by a Logarithmic distribution* with p. g. f.

$$g_2(z) = 1 - \alpha \log(q - pz) \qquad \begin{array}{l} p > 0 \\ \alpha > 0 \\ q = 1 + p \end{array} \tag{7}$$

Now, the number of larvae at the random location may be due to 0, 1, 2, ... egg-masses. Consequently, the over-all distribution of larvae will have p. g. f.

$$g(z) = \sum_{r=0}^{\infty} P_r\{g_2(z)\}^r = g_1\{g_2(z)\} \tag{8}$$

which, in the present instance, reduces to

$$g(z) = e^{-\alpha \log(q-pz)} = (g - pz)^{-\alpha} \tag{9}$$

the p. g. f. of a Negative Binomial distribution. On the basis of this model the Negative Binomial may be regarded as a generalized Poisson distribution

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*This Logarithmic distribution is more general than that considered by Fisher, Corbett, and Williams (1943) or by Jones, Mollison, and Quenouille (1948) in that it permits a positive probability for the occurrence of zero counts. When $1 - \alpha \log q = 0$ it reduces to the more specialized Logarithmic distribution.

(cf. Quenouille (1949)). A formal relation between certain families of compound and generalized distributions will be considered in section 3.

3. A formal relation between some compound and generalized distributions.

For convenience we employ the definitions and notation employed by Gurland (1957).

Definition 1 Compound distribution

Let the random variable X_1 have the distribution function $F_1(x_1|\theta)$ for a given value of the variable X_1 and of the parameter θ. Suppose now that θ is regarded as a random variable X_2, say, with distribution function F_2. Denote by $X_1 \wedge X_2$ the random variable with distribution function F given by

$$F(x_1) = \int_D F_1(x_1|cx_2)\, dF_2(x_2) \qquad (10)$$

for each value of X_1, where D is the domain of F_2. Here c is a constant which is arbitrary. (Values of c for which (10) is not a distribution function are excluded). The random variable $X_1 \wedge X_2$ (uniquely defined here apart from the constant c) is called a compound X_1 variable with respect to the "compounder" X_2.

In the example of 2.1, X_1 is a Poisson random variable with mean λ and X_2 is a Gamma random variable with probability density given by (3).

The constant c was taken as unity, but in this example there is no loss of generality because (4) would have become, with c in place of unity, $[1 - \frac{c}{\alpha}(z-1)]^{-\beta}$; and we would then define $p = c/\alpha$ instead of $1/\alpha$.

Definition 2 Equivalent distributions

Suppose the random variables X_1, X_2 have distribution functions $F_1(x/\alpha)$, $F_2(x/\beta)$ respectively. α and/or β may be multi-dimensional. If for each α there exists some β and for each β there exists some α such that $F_1(x/\alpha) = F_2(x/\beta)$ whatever be x, the random variables X_1, X_2 are said to be equivalent, and we write $X_1 \sim X_2$.

It is often convenient to represent a random variable by the name of its corresponding distribution. Thus, in the case of the compound Poisson considered in section 2.1 we might write

$$\text{Poisson} \wedge \text{Gamma} \sim \text{Negative Binomial} \tag{10}$$

It may happen as in several cases considered below that the initial distribution being compounded may have several parameters but only a particular one of them is regarded as a random variable. In such cases the notation $X_1 \wedge X_2$ as employed in (10) might become ambiguous; for these cases the notation will be modified as required. In the example above represented by (10) there is no ambiguity since the Poisson has only one parameter, namely, the mean.

Definition 3 Generalized distribution

Let the random variables X_1, X_2 have p.g.f.'s $g_1(z)$, $g_2(z)$ respectively. Denote by $X_1 \vee X_2$ the random variable with p.g.f. $g_1(g_2(z))$. Then $X_1 \vee X_2$ is called a generalized X_1 variable with respect to the "generalizer" X_2.

Theorem

Let X_1 be a random variable with p.g.f. $[h(z)]^{\theta}$ where θ is a given parameter. Suppose now θ is regarded as a random variable X_2, say, with distribution function F_2 and p.g.f. g_2. Then, whatever be X_2,

$$X_1 \wedge X_2 \sim X_2 \vee X_1 \tag{11}$$

assuming the p.g.f. of these random variables exists.

Proof

The proof follows immediately from the definition of compound and generalized distributions. In fact, the p.g.f. of $X_1 \wedge X_2$ is given by

$$\int_D [h(z)]^{cx} dF_2(x)$$

while that of $X_2 \vee X_1$ is given by

$$g_2\{g_1(z)\} = \int_D [h(z)]^{\theta x}\, dF_2(x) \quad .$$

These are, of course, equal, when $c = \theta$.

It is interesting to note the role of the constant c introduced in the definition of compound random variable.

As an example of applying the above theorem let X_1 and X_2 both be Poisson random variables. Then

$$\text{Poisson} \wedge \text{Poisson} \sim \text{Poisson v Poisson} \quad . \tag{12}$$

This distribution is called the Neyman Type A (cf. Neyman, 1939), and may be interpreted both as a compound Poisson and as a generalized Poisson, as was pointed out by Feller (1943).

It should be noted both in the theorem and in the above definitions that the random variables X_1, X_2 need not be discrete. For X_1 the p.g.f. is Ez^{X_1} and likewise, of course, for X_2. The following example illustrates the point.

$$\text{Poisson} \wedge \text{Gamma} \sim \text{Gamma v Poisson} \quad . \tag{13}$$

To verify (13) we note that Poisson $\wedge$ Gamma is equivalent to a Negative Binomial. It suffices, therefore, to show that Gamma v Poisson is also equivalent to a Negative Binomial. Now the moment generating function Ee^{tX} of the Gamma random variable X with probability density given by (3) is

$$\left(1 - \frac{t}{\alpha}\right)^{-\beta} . \tag{14}$$

Replacing e^t by z yields the p.g.f.

$$\left(1 - \frac{\log z}{\alpha}\right)^{-\beta} . \tag{15}$$

If the p.g.f. $e^{\lambda(z-1)}$ of the Poisson is substituted for z in (15) we obtain

$$\left[1 - \frac{\lambda}{\alpha}(z-1)\right]^{-\beta}$$

which corresponds to a Negative Binomial as required.

Let us next consider examples of compounding a distribution which involves more than one parameter. Take, for instance, a Negative Binomial with p.g.f. $(q - pz)^{-k}$. For brevity we shall refer to this distribution as *Pascal (k, p). The above theorem and relation (11) apply if the index parameter k is regarded as the random variable X_2. Taking X_2 to be a Poisson and a Gamma random variable respectively yields the following relations

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* Although the term "Pascal distribution" commonly refers to the particular case of a Negative Binomial distribution with index parameter k an integer, we employ the same terminology for the Negative Binomial for convenience in writing. (cf. Gurland (1959) Katti and Gurland (1962))

$$\text{Pascal}\,(k, p) \underset{k}{\wedge} \text{Poisson} \sim \text{Poisson} \vee \text{Pascal} \tag{16}$$

$$\text{Pascal}\,(k, p) \underset{k}{\wedge} \text{Gamma} \sim \text{Gamma} \vee \text{Pascal} \tag{17}$$

It should be noted the letter k is inserted below the symbol "$\wedge$" to obviate the possible ambiguity mentioned earlier.

The examples in sections 2.1 and 2.3 exhibiting the Pascal distribution as a compound Poisson and generalized Poisson, respectively, can be expressed symbolically as

$$\text{Poisson} \wedge \text{Gamma} \sim \text{Poisson} \vee \text{Logarithmic}\ . \tag{18}$$

It was shown by Gurland (1957) that this relation can be extended. Thus,

$$(\text{Poisson} \wedge \text{Gamma}) \wedge \text{Gamma} \sim (\text{Poisson} \vee \text{Logarithmic}) \vee \text{Logarithmic} \tag{19}$$

that is

$$\text{Pascal} \wedge \text{Gamma} \sim \text{Pascal} \vee \text{Logarithmic}\ . \tag{20}$$

This extension can, in fact, be carried out any number of times. The next step, for example, would be

$$(\text{Pascal} \wedge \text{Gamma}) \wedge \text{Gamma} \sim (\text{Pascal} \vee \text{Logarithmic}) \vee \text{Logarithmic} \tag{21}$$

and so on.

4. A generalization of Neyman's class of contagious distributions

Let us consider the example in section 2.3 in more detail and in a modified form. As before, let the probability that exactly r egg-masses are represented on a randomly selected location be given by a Poisson distribution

$$P_r = e^{-\lambda_1} \lambda_1^r / r! \tag{22}$$

Before we were interested merely in the number of larvae which move from an egg-mass to a particular location. In the present instance we are also interested in the number of survivors in an egg-mass, that is, the number of larvae that hatch out. Suppose the number of survivors in an egg-mass is a Poisson random variable with mean λ, say. That is, the probability that there are exactly n survivors in an egg-mass is given by

$$e^{-\lambda} \lambda^n / n! \tag{23}$$

Suppose that in a particular egg-mass there are n survivors. The probability that exactly s of them will be found at a particular location will be assumed to be

$$\binom{n}{s} p^s (1-p)^{n-s} \tag{24}$$

which corresponds to a Binomial distribution with parameters n, p.

A straightforward application of the notions of compound and generalized distributions discussed in sections 2 and 3 yields as the p.g.f. of the distribution of larvae

$$e^{\lambda_1[g(z)-1]} \tag{25}$$

where $g(z)$ is the p.g.f. of the Binomial distribution in (24) compounded with the Poisson distribution in (23). A simple argument utilizing the relation

$$\text{Binomial } (n, p) \underset{n}{\wedge} \text{Poisson} \sim \text{Poisson v Binomial}$$

shows that

$$g(z) = e^{\lambda p(z-1)} \tag{26}$$

which corresponds to a Poisson. Consequently, the resulting distribution given by (25) is a Neyman Type A .

As a first step in extending this family of distributions suppose the parameter p in (24) may (more realistically) be regarded as a random variable, following, say, a Beta distribution with probability density

$$\frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \qquad \begin{array}{l} 0 < x < 1 \\ \alpha > 0; \quad \beta > 0 \end{array} \tag{27}$$

On compounding the distribution in (26) with this Beta distribution we obtain the p.g.f. $g_1(z)$, say, where

$$g_1(z) = \frac{1}{B(\alpha, \beta)} \int_0^1 e^{\lambda x(z-1)} x^{\alpha-1}(1-x)^{\beta-1}\, dx = {}_1F_1\{\alpha,\ \alpha+\beta,\ \lambda(z-1)\} \tag{28}$$

and where ${}_1F_1$ is the well-known confluent hypergeometric function. For convenience let us refer to the distribution in (28) as Type H_1. Then the distribution of larvae is a generalized Poisson represented by

$$\text{Poisson} \vee \text{Type } H_1 \tag{29}$$

as obtained by Gurland (1958). If in (28) we set $\alpha = 1$, the family (29) reduces to that of Beall and Rescia (1953).

As a further step in extending Neyman's family of distributions the parameter λ in (23) may also be regarded as a random variable. This is a realistic consideration because different egg-masses would conceivably be associated with different probabilities of survival. If we assume a Gamma distribution for λ, then (23) becomes a Pascal distribution with p.g.f., say, $(q_1 - p_1 z)^{-k_1}$. Treating p in (24) as a random variable as before, the distribution of larvae becomes a generalized Poisson with p.g.f.

$$e^{\lambda_1[g_2(z)-1]}$$

where

$$g_2(z) = \frac{1}{B(\alpha, \beta)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[1-p_1 x(z-1)]^{k_1}}\, dx = {}_2F_1\{k_1, \alpha, \alpha+\beta, p_1(z-1)\} \ . \quad (30)$$

If, for convenience, we refer to the distribution corresponding to $g_2(z)$ as Type H_2, then the distribution of larvae may be represented by

$$\text{Poisson v Type } H_2 \ . \quad (31)$$

If, in addition to the above compounding we also allow the parameter λ_1 in (22) to follow a Gamma distribution, the distribution of egg-masses becomes a Pascal. In analogy with (29) and (31) we obtain two more families of distributions represented by

$$\text{Pascal v Type } H_1 \quad (32)$$

$$\text{Pascal v Type } H_2 \quad (33)$$

respectively.

As some of these general families contain many parameters and are not particularly simple to work with it would be interesting to examine their characteristics in the hope of finding simpler families which might be similar in shape. Some results along these lines are considered in section 5 .

5. Skewness and kurtosis of some families of distributions

Among the usual characteristics of interest in assessing the shape of a distribution are the skewness and kurtosis. These are measured by $\mu_3/\mu_2^{3/2}$, μ_4/μ_2^2, respectively, where μ_2, μ_3, μ_4 are central moments of the orders indicated by the subscripts. To standardize the distributions under comparison in some reasonable sense, we have reparametrized them to have the same mean kp and the same variance $kp(1 + p)$ as the Negative Binomial. This is suggested by a similar comparison made by Anscombe (1950) in the case of a few two-parameter families of distributions he compared with the Negative Binomial.

As measures of skewness and kurtosis we have also employed the same quantities $\kappa_{(3)}/kp^3$, $\kappa_{(4)}/kp^4$ as Anscombe (op. cit.), where $\kappa_{(3)}$ and $\kappa_{(4)}$ are the third and fourth factorial cumulants. For the distributions we have considered these measures are particularly convenient both from the standpoint of calculation and from the fact the final measures obtained do not involve the parameters k, p .

A note of caution should be made, however, in the use of the above quantities as measures of skewness and kurtosis. Since

$$\mu_3 = \kappa_{(3)} + \text{ a function involving only the first two moments}$$

$$\mu_4 = \kappa_{(4)} + 6\kappa_{(3)} + \text{ a function involving only the first two moments}$$

and the first two moments of all the distributions under comparison are the same it follows that when $\kappa_{(3)}/kp^3$ and $\kappa_{(4)}/kp^4$ are both increasing or both decreasing then the distributions can, in fact, be ordered according to skewness and kurtosis. For all the two-parameter families appearing in Table 1 this is actually the case. For those families containing more than two parameters and involving the Type H_1 or Type H_2 distributions there are some values of the parameters for which the above quantities involving factorial cumulants increase or decrease in opposite directions. The interval between minimum and maximum values, however, is of some value in the comparison of the shapes of the various distributions in Table 1. Each pair of numbers in the table enclosed in parentheses indicates such an interval.

As a further explanation of the distributions appearing in Table 1, the Neyman B and Neyman C are special cases of the family (29) with $\alpha = 1$ and $\beta = 1, 2$ respectively in (27). The Polya-Aeppli distribution is also a special case of the above family with $\alpha = 1$ and $\beta = \infty$. (cf. Gurland (1958)). The Polya-Aeppli distribution can also be defined formally as a special case of the Poisson v Pascal with p.g.f. $e^{\lambda[(q-pz)^{-1}-1]}$.

TABLE 1

Measure of skewness and kurtosis for some distributions with the same first two moments kp, kp(1 + p)

Distribution	$\frac{\kappa_{(3)}}{kp^3}$	$\frac{\kappa_{(4)}}{kp^4}$
Poisson v Binomial	(0, 1)	(0, 1)
Neyman A	1	1
Neyman B	9/8	27/20
Neyman C	6/5	8/5
Polya-Aeppli	3/2	3
Pascal	2	6
Pascal ∧ Gamma	(1.75, 2)	(4.373, 6)
Poisson v Pascal	(1, 2)	(1, 6)
Pascal v Poisson	(1, 2)	(1, 6)
Pascal v Pascal	(1, 2)	(1, 6)
Poisson v H_1	(1, 2)	(1, 6)
Pascal v H_1	(1, 2)	(1, 6)
Poisson v H_2	(1, 4)	(1, 36)

It is apparent from Table 1 that for all the two-parameter families under consideration the skewness and kurtosis are both increasing. From the Neyman A on through to the Pascal there is a range (1, 2) for the skewness measure and a range (1, 6) for the kurtosis measure. It is particularly interesting that for the Poisson v Pascal, Pascal v Poisson, Pascal v Pascal, Poisson v H_1, and Pascal v H_1 the range between minimum and maximum for the skewness measure is also (1, 2) and for the kurtosis measure is also (1, 6). Note that the Poisson v Pascal and Pascal v Poisson are three-parameter families whereas the Pascal v Pascal, Poisson v H_1 involve four parameters, the Pascal v H_1, Poisson v H_2 involve five parameters.

As the Poisson v Pascal and Pascal v Poisson are simpler families than those involving more parameters their flexibility of shape is a recommendation in favor of their use. Of these two distributions the Poisson v Pascal lends itself to simpler computation of the probabilities and extimation of the parameters required in the fitting of the distribution to observed data.

It is also evident from Table 1 that the Poisson v Binomial covers the range of skewness (0, 1) and the range of kurtosis (0, 1). As the corresponding ranges for the Poisson v Pascal are (1, 2) and (1, 6), this shows that these relatively simple three-parameter families, the Poisson v Binomial and the Poisson v Pascal cover a wide range of possible shapes. Methods of estimating the parameters and computing the probabilities in these distributions appear in a number of recent papers. Shumway and Gurland (1960), (1961), Katti and Gurland (1961), (1962a).

6. Considerations in the choice of a family of contagious distributions

From the preceding sections it is evident that many forms of compound and generalized distributions are possible. As some of these distributions are simpler than others, yet are meaningful biologically and do not suffer seriously in loss of flexibility, the following three criteria might be suggested as important in the choice of an appropriate family

(i) Simplicity

(ii) Flexibility

(iii) Meaningful parameters

The Negative Binomial is one of the most widely used discrete distributions because it is relatively simple and is very convenient computationally although the estimation of the parameters is rather tedious if the method of maximum likelihood is employed (cf. Fisher(1953) Bliss (1953)).

The Neyman Type A distribution, a two-parameter family, is also widely used (cf. Beall(1940) Evans(1953))but it is not as convenient in computing probabilities as is the Negative Binomial. Methods have been devised for simplifying these computations (cf. Douglas (1955)).The estimation of the parameters by maximum likelihood is also tedious, but alternative methods which are simpler and retain high efficiency have been suggested both for the Negative Binomial and the Neyman Type A by Katti and Gurland (1962b).

If none of the relatively simple distributions such as the Poisson, Negative Binomial, Neyman Type A is appropriate then one of the the three-parameter families suggested in section 5 might be utilized. On the basis of only a few isolated experiments it is, of course, not possible to distinguish effectively between competing distributions, in which case the simpler ones, if they provide a good fit, are to be preferred. On the other hand, if many experiments are carried out in the same classes of situations, and if there is ample evidence that none of the simple distributions is appropriate, then a more flexible distribution such as the Poisson v Pascal, say, might be tried.

The Poisson v Pascal affords an attractive alternative because it is also relatively simple (almost as easy to work with as the Neyman Type A) and because it subsumes the Negative Binomial, the Neyman Type A, and the Poisson as limiting cases (cf. Katti and Gurland (1961)). Specifically, let a Poisson v Pascal have p.g.f. $g(z) = e^{\lambda[(q-pz)^{-k}-1]}$. Table 2 gives the limiting form of $g(z)$ for different passages to the limit.

TABLE 2

Some limiting forms of the Poisson v Pascal distribution

No.	Limits taken	Limiting p.g.f.	Name of limiting distribution
1	$k \to \infty$, $p \to 0$ $pk = \lambda_1$	$e^{\lambda[e^{\lambda_1(z-1)}-1]}$	Neyman Type A
2	$k \to 0$, $\lambda \to \infty$ $\lambda k = k_1$	$(q-pz)^{-k_1}$	Negative Binomial
3	$p \to 0$, $\lambda \to \infty$ $\lambda kp = \lambda_1$	$e^{\lambda_1(z-1)}$	Poisson

Some methods for simplifying the computation of the probabilities and for obtaining the maximum likelihood estimates of the parameters in a Poisson v Pascal distribution are given by Shumway and Gurland (1961). Estimation of the parameters in this distribution by the technique of minimum chi-square is considered by Katti and Gurland (1961). In Table 3, taken from this paper, we see the results of fitting a Poisson v Pascal and a Polya-Aeppli to some data of Beall and Rescia (1953).

TABLE 3

Fit of the observed frequency of Lespedeza Capitata from Table V of Beall-Rescia (1953)

Plants	Observed Frequency	Expected frequency due to Poisson v Pascal (Method of moments)	Expected frequency as in Beall-Rescia (1953)
0	7178	7185.0	7217.6
1	286	276.0	218.6
2	93	94.5	105.5
3	40	41.5	50.9
4	24	20.2	24.5
5	7	10.4	11.8
6	5	5.6	5.7
7	1	3.1	2.8
8	2	1.7	1.3
9	1	1.0	.6
10	2	.6	.3
11+	1	.3	.4
χ^2		9.58	42.97
Degrees of freedom		8	9

It is evident from the χ^2 values at the foot of Table 3 that the Poisson v Pascal definitely provides a much closer fit. This is not surprising because of the much greater flexibility of the Poisson v Pascal.

For a lower range of skewness and kurtosis the information in Table 1 suggest the use of the Poisson v Binomial distribution. From the form of its p.g.f. $g(z) = e^{\lambda[(q+pz)^n - 1]}$ it is evident this distribution converges rather quickly to the Neyman Type A distribution as $n \to \infty$, $p \to 0$ with np constant. For small values of n, however, it may be quite useful, and has been applied by Mc Guire et al. (1956) and Sprott(1958).

TABLE 4

Fit of the observed frequency of Pyrausta Nubilalis from Distribution 6 of Mc Guire et al. (1957)

Corn Borers	Observed Frequency	Expected frequency due to Poisson v Binomial (n = 2)	Expected frequency due to Poisson v Binomial (n = 3)
0	907	906.18	907.66
1	275	276.69	277.24
2	88	89.92	86.50
3	23	18.86	20.14
4	3	4.35	3.23
χ^2		1.39	0.47
Degrees of freedom		2	2

In Table 4 are shown the results of fitting a Poisson v Binomial to some data of Mc Guire et al. (1957) by the method of maximum likelihood. This table is partially reproduced from Shumway and Gurland (1961). It is quite evident that a good fit is provided in the case $n = 2$ and an even better fit in the case $n = 3$. Techniques for estimating the parameters of a Poisson v Binomial based on minimum chi-square have been developed by Katti and Gurland (1962 a).

7. Conclusion

One might ask what is the purpose of fitting data by discrete distributions such as those considered here. Apropos of this question it is interesting that in the application of most standard statistical techniques based on the Normal distribution a test of fit is not usually performed. This may be due to a wide experience of a good fit by the Normal distribution or to the property of robustness (cf. Box and Anderson (1955)) enjoyed by many tests which are based on a Normal population but in applying which the data is actually from a non-Normal population.

In the case of data from a discrete distribution many underlying forms are possible and the fittings based on these forms may be quite different. A knowledge of the underlying distribution makes it at least theoretically possible to construct tests and estimate parameters for the purpose of making statistical inference.

It is also important for the distributions fitted to biological data to be based on models which have a reasonable biological meaning. The compound and generalized distributions, including the Negative Binomial, Neyman Type A, Poisson v Pascal, and many others, afford interesting possibilities of such distributions, because they provide a simple mechanism for explaining the "clumpiness" which is so characteristic of much biological data.

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